A note on the weighted uniform distribution mod 1

by

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Abstract

M. Tsuji[3] generalized the result of H. Weyl[4] concerning the theory of uniformly distribution to the case of weighted means. And the case of the weighted uniformly convergence was generalized by the auther's paper[1]. T. Kano[2] showed the necessary and sufficient condition for a sequence to be weighted uniformly distributed mod 1 when a weight and a sequence satisfy some conditions.

It is our aim in this paper to extend Kano's result to the case of weighted uniformly convergence.

Now we shall begin with two key definitions:

Let f be a complex-valued continuous function on $(-\infty, +\infty)$ with period 1.

Definition 1. The sequence (x_n) is said to be (M, λ_n)-uniformly distributed mod 1 (abbreviated (M, λ_n)-u. d. (mod 1) if

$$\lim_{n\to\infty} \frac{1}{\Lambda_n}\sum_{k=1}^{n} \lambda_k f(x_k) = \int_0^1 f(x)dx$$

where

$$\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq \cdots > 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Definition 2. The sequence (x_n) is said to be (M, λ_n)-well distributed mod 1 (abbreviated (M, λ_n)-w. d. (mod 1) if

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^k}\sum_{\nu=k+1}^{k+n} \lambda_\nu f(x_\nu) = \int_0^1 f(x)dx \qquad \text{uniformly in } k=0,1,2,\cdots,$$

where

$$\Lambda_n^k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_{k+n}, \quad \lambda_{k+1} \geqq \lambda_{k+2} \geqq \cdots \geqq \lambda_{k+n} \geqq \cdots 0. \quad \sum_{n=1}^{\infty} \lambda_{k+n} = \infty.$$

Kano[2] proved the following theorem:

Let $\lambda_n = \lambda(n)$, where $\lambda(t) > 0 (t \geq 1)$ is a non-increasing function of class $C^1[1, \infty)$

such that $\int_1^t \lambda(u)du\to\infty$, and let g(t), defined for $t\geq 1$, be such that

(a) $g(t)\in C^2[1,\infty)$,

(b) $g'(t)\to 0$ $(t\to\infty)$,

(c) $\lim_{t\to\infty}\frac{1}{\lambda(t)}\cdot\frac{d}{dt}\left(\frac{\lambda(t)}{g'(t)}\right)$ exists.

Then $\{g(n)\}$ $(n=1,2,\cdots)$ is (M,λ_n)-uniformly distributed mod 1 if and only if

$$Q(t)\overset{\text{def}}{=}\frac{g'(t)}{\lambda(t)}\int_1^t \lambda(u)du\to\pm\infty \qquad (t\to\infty).$$

We shall state the following extension of above Theorem.

Theorem 1. Let $\lambda(t)>0(t\geq 1)$ be a non-increasing function of class $C^1[1,\infty)$ such that $\int_{k+1}^{k+n}\lambda(t)dt\to\infty$ uniformly in k and let g(t), defined for $t\geq 1$, be such that

(a) $g(t)\in C^2[1,\infty)$,

(b) $g'(t)\to 0$,

(c) $\lim_{t\to\infty}\frac{1}{\lambda(k+t)}\cdot\frac{d}{dt}\left(\frac{\lambda(t+k)}{g'(t+k)}\right)$.

exists, and the convergence is uniformly in $k=0,1,2,\cdots,$.

Then $\{g(n)\}$ is (M,λ_n)-w. d. (mod 1) if and only if

$$Q_k(t)=\frac{g'(t+k)}{\lambda(t+k)}\int_{k+1}^{k+t}\lambda(u)du\to\pm\infty.$$

as $t\to\infty$, uniformly in k.

Lemma. Let g(t) be such a function defined for $t\geq 1$ that

(i) $g(t)\in C^2[1,\infty)$

(ii) $g'(t)\to 0$, not monotone,

(iii) $\lim_{t\to\infty}\frac{d}{dt}\left(\frac{1}{g'(t)}\right)$ exists.

then

$$\lim_{t\to\infty}\frac{d}{dt}\left(\frac{1}{g'(t)}\right)=0.$$

Proof. First we assume $\liminf>0$. From the assumption, there is a continuous function h(t) such that

$$\frac{d}{dt}\left(\frac{1}{g'(t)}\right)=h(t)>0$$

hence

$$g''(t)=\frac{-h(t)}{\left\{\int_{t_0}^t h(y)dy\right\}^2}.$$

By the assumption (ii), the sign of $g''(t)$ changes infinitely many times as t tends to infinite. This contradicts the assumption of h(t). Therefore $\liminf\leq 0$.

In the same way, it would follow that limsup$\geqq 0$. The proof is complete. (q. e. d.)

Proof of Theorem 1. From the above lemma, the limit of condition (c) is 0. Hence, by easy computation, $Q_k(t)$ tends to $\pm\infty$ uniformly in $k=0,1,2,\cdots\cdots$,. Next we shall show the sufficiency. We begin with the case that $Q_k(t)\to\alpha\neq 0$ as $t\to\infty$. by (c)

$$(1)\quad \frac{d}{dt}\left(\frac{\lambda(t+k)}{g'(t+k)}\right)=\frac{1}{\alpha}\lambda(t+k)+o(\lambda(t+k)) \qquad (t\to\infty).$$

By Euler summation formula

$$(2)\quad \sum_{\nu=k+1}^{k+n}\lambda(\nu)e^{2\pi img(\nu)}=\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt+o\left(\int_{k+1}^{k+n}\lambda(t)dt\right).$$

If we assume that $\{g(n)\}$ is (M, λ_n)-w. d. (mod 1) we have by (2)

$$\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt=o\left(\int_{k+1}^{k+n}\lambda(t)dt\right).$$

On the other hand integration by parts shows

$$(3)\quad (2\pi im)\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt=\frac{\lambda(k+n)}{g'(k+n)}e^{2\pi img(k+n)}-\frac{\lambda(k+1)}{g'(k+1)}e^{2\pi img(k+1)}$$
$$-\int_{k+1}^{k+n}e^{2\pi img(t)}\frac{d}{dt}\left(\frac{\lambda(t)}{g'(t)}\right)dt.$$

By (1), we obtain

$$\int_{k+1}^{k+n}e^{2\pi img(t)}\frac{d}{dt}\left(\frac{\lambda(t)}{g'(t)}\right)dt=\frac{1}{\alpha}\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt+o\left(\int_{k+1}^{k+n}\lambda(t)dt\right).$$

From (3)

$$e^{2\pi img(k+n)}=\frac{g'(k+n)}{\lambda(k+n)}\Big[(2\pi im)\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt+\frac{\lambda(k+1)}{g'(k+1)}e^{2\pi img(k+1)}$$
$$+\int_{k+1}^{k+n}e^{2\pi img(t)}\frac{d}{dt}\left(\frac{\lambda(t)}{g'(t)}\right)dt\Big]$$
$$=\frac{g'(k+n)}{\lambda(k+n)}\Big[\frac{\lambda(k+1)}{g'(k+1)}e^{2\pi img(k+1)}+o\left(\int_{k+1}^{k+n}\lambda(t)dt\right)\Big]=o(1).$$

since assumption of $\lambda(t)$ and $Q_k(t)\to\alpha\neq 0$. Hence we obtain

$$e^{2\pi img(k+n)}=o(1).$$

which is a contradiction. Next we assume $Q_k(t)\to\pm\infty$ uniformly in k. By Euler summation formula, we obtain (2). Hence from (c) and assumption of $\lambda(t)$,

$$2\pi im\int_{k+1}^{k+n}\lambda(t)e^{2\pi img(t)}dt=o\left(\int_{k+1}^{k+n}\lambda(u)du\right) \text{ uniformly in } k$$

Lastly we prove that $\{g(n)\}$ is not (M, λ_n)-u. d. (mod 1) when $Q_k(t)\to 0$. This is a conseequence of the following theorem. (q. e. d.)

Theorem 2. Let $g(t)$ be differentiable for $t\geqq 1$ and let $\lambda(t)$ be the same function in Theorem 1. If

$$\frac{g'(t+k)}{\lambda(t+k)}\int_{1+k}^{t+k}\lambda(u)du\to 0 \qquad (t\to\infty)$$

then $\{g(n)\}$ is not (M, λ_n)-w. d. (mod 1).

Proof. We assume

$$\frac{g'(t+k)}{\lambda(t+k)}\int_{1+k}^{t+k}\lambda(u)du \to 0,\ t\to\infty,$$

and further

$$(4)\quad \sum_{\nu=k+1}^{k+n}\lambda_\nu e^{2\pi i m g(\nu)} = o\left(\sum_{\nu=k+1}^{k+n}\lambda_\nu\right).$$

when $n\to\infty$. Then we have

$$\frac{g'(t+k)}{\lambda(t+k)}\sum_{\nu=1+k}^{[t]+k}\lambda_\nu \to 0,$$

$$(5)\quad g'(t+k) = o\left(\frac{\lambda(t+k)}{\sum_{\nu-1+k}^{[t]+k}\lambda_\nu}\right).$$

On the oter hand partial summation formula shows

$$(6)\quad \mathrm{Re}\sum_{\nu=k+1}^{k+n}\lambda_\nu e^{2\pi i m g(\nu)} = \sum_{\nu=k+1}^{k+n-1}(C_\nu - C_{\nu+1})\Lambda_\nu^k + C_{k+n}\Lambda_n^k$$

where

$$C_\nu = \cos(2\pi m g(\nu))$$

$$(7)\quad \left|\sum_{\nu=k+1}^{k+n-1}(C_\nu - C_{\nu+1})\Lambda_\nu^k\right| \geq \sum_{\nu=k+1}^{k+n-1}\left|C_\nu - C_{\nu+1}\right|\Lambda_\nu^k = o\left(\Lambda_n^k\right).$$

since (5), $\lambda(t)\downarrow$ and $\Lambda_k{}^n\uparrow\infty$. Therefore from (4), (6) and (7)

$$C_{k+n} = \cos(2\pi m g(k+n)) = o(1).$$

and similary

$$\sin(2\pi m g(k+n)) = o(1).$$

that is, we obtain, as $t\to\infty$

$$e^{2\pi i m g(k+n)} = o(1).$$

This is a contradiction. (q. e. d.)

References

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