On the Wiener-Schoenberg Theorem for the (M, λ)-well continuous distribution mod 1.

by

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Abstract

The author studied in [1] the conditions that the sequence has (M, λ_n)-weighted uniform distribution function mod 1 g(x), that g(x) is continuous and that g(x) is absolutely continuous.

In this paper, we shall prove the analogue results for the continuous distribution function [2; Chap 1. § 9].

Let $\lambda(t)$ defined on $[0, \infty)$ be a positive monotone decreasing and $\Lambda(T)=\int_0^T \lambda(t)dt$ tend to infinity as T.

Definition 1. *The Lebesgue-measurable function $f(t)$ defined on $[0, \infty)$ is said to have (M, λ)-asymptotic continuous distribution function mod* 1 (*abbreviated* (M, λ)-*a. c. d. f.* (*mod* 1)) $g(x)$ *if for each real-valued continuous function w on* $[0, 1]$,

$$\lim_{T\to\infty}\frac{1}{\Lambda(T)}\int_0^T \lambda(t)w(\{f(t)\})dt=\int_0^1 w(x)dg(x),$$

where $\{t\}$ is the fractional part of t.

Definition 2. *The Lebesgue-measurable function $f(t)$ defined on $[0, \infty)$ is said to have (M, λ)-asymptotic well continuous distribution function mod* 1 (*abbreviated* (M, λ)-*a. w. c. d. f.* (*mod* 1)) $g(x)$ *if for each real valued continuous function w on* $[0, 1]$,

$$\lim_{T\to\infty}\frac{1}{\Lambda(T, k)}\int_{0+k}^{T+k} \lambda(t)w(\{f(t)\})dt=\int_0^1 w(x)dg(x) \text{ uniformly in } k\in[0, \infty),$$

where $\Lambda(T, k)=\int_{0+k}^{T+k}\lambda(t)dt$.

Now we shall state the results.

Theorem 1. *The Lebesgue-measurable function $f(t)$ defined on $[0, \infty)$ is (M, λ)-a. w. c. d. f. (mod* 1) $g(x)$ *if and only if for all* $h\in \mathbf{Z}$

(1) $$\lim_{T\to\infty}\frac{1}{\Lambda(T,k)}\int_{0+k}^{T+k}\lambda(t)e^{2\pi ihf(t)}dt=\alpha_h$$

exists uniformly in $k\in[0,\infty)$ *and*

(2) $$\alpha_h=\int_0^1 e^{2\pi ihx}dg(x).$$

Theorem 2. *The function* $(f(t))$ *has a continuous* (M,λ)*-a. w. c. d. f. (mod 1) if and only if for every positive integer h the limit* (1) *exists uniformly in* $k\in[0,\infty)$ *and, in addition*

(3) $$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\alpha_h|^2=0.$$

Theorem 3. *Let the function* $f(t)$ *have* (M,λ)*-a. w. c. d. f. (mod 1)* $g(x)$ *Then* $g(x)$ *is absolutely continuous and* $g'(x)\in L^2(0,1)$ *if and only if for all* $h\in\mathbf{Z}$

(4) $$\alpha_h=\lim_{T\to\infty}\frac{1}{\Lambda(T,k)}\int_{0+k}^{T+k}\lambda(t)e^{2\pi ihf(t)}dt,$$

exists uniformly in $k\in[0,\infty)$ *and, in addition*

(5) $$\sum_{h=-\infty}^{\infty}|\alpha_h|^2<+\infty.$$

Taking k=0, we obtain the following results easily.

Corollary 1. *The Lebesgue-measurable function* $f(t)$ *defined on* $[0,\infty)$ *has* (M,λ)*-a. c. d. f. (mod 1)* $g(x)$ *if and only if for all* $h\in\mathbf{Z}$

(6) $$\lim_{T\to\infty}\frac{1}{\Lambda(T)}\int_0^T\lambda(t)e^{2\pi ihf(t)}dt=\alpha_h$$

Corollary 2. *The function* $(f(t))$ *has a continuous* (M,λ)*-a. c. d. f. (mod 1) if and only if for every positive integer h the limit* (6) *exists and, in addition*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\alpha_h|^2=0.$$

Corollary 3. *Let the function* $f(t)$ *have* (M,λ)*-a. c. d. f. (mod 1)* $g(x)$. *Then* $g(x)$ *is absolutely continuous and* $g'(x)\in L^2(0,1)$ *if and only if for all* $h\in\mathbf{Z}$

$$\alpha_h=\lim_{T\to\infty}\frac{1}{\Lambda(T)}\int_0^T\lambda(t)e^{2\pi ihf(t)}dt=\int_0^1 e^{2\pi ihx}dg(x),$$

exists and, in addition

$$\sum_{h=-\infty}^{\infty} |\alpha_h|^2 < +\infty.$$

The proof of Theorem 1, 2 and 3 runs along the same lines as [1]. We shall prove above results.

Proof of 1. The necessity follows from the fact that the function exp (2πihx) is continuous on ($-\infty$, ∞) with period 1. Now assume that (f(t)) satisfies (1), and p(x) is a continuous function on [0, 1]. By Weiestrass' approximation theorem, there exists a complex trigonometric polynomial P(x), that is, a finite linear combination of functions like exp(2πimx) ($m \in \boldsymbol{Z}$) such that for any positive ε, we have

$$\sup_{0 \leq x \leq 1} |p(x) - P(x)| < \varepsilon.$$

Thus, for n sufficient large, using triangle inequality,

$$\left| \frac{1}{\Lambda(T, k)} \int_{0+k}^{T+k} \lambda(t) p(\{f(t)\}) dt - \int_0^1 p(x) dg(x) \right|$$
$$\leq 2\varepsilon + \left| \frac{1}{\Lambda(T, k)} \int_{0+k}^{T+k} \lambda(t) p(\{f(t)\}) dt - \int_0^1 p(x) dg(x) \right| < 3\varepsilon,$$

since the last term, as $n \to \infty$, tends to zero uniformly in k by virture of (1).

(q. e. d.).

Proof of 2. The existence of the limit (1) is necessary. Next we prove that if for (f(t)) we have

$$\alpha_h = \lim_{T \to \infty} \frac{1}{\Lambda(T, k)} \int_{0+k}^{T+k} \lambda(t) e^{2\pi i h f(t)} dt = \int_0^1 e^{2\pi i h x} dg(x).$$

uniformly in $k \in [0, \infty)$ for all positive integers h, then g(x) is continuous if and only if (3) holds. Because we have

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\alpha_h|^2 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \alpha_h \bar{\alpha}_h = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \int_0^1 \int_0^1 e^{2\pi i h (x-y)} dg(x) dy(y)$$

$$= \int_0^1 \int_0^1 \left(\text{iim}_{T \to \infty} \frac{1}{H} \sum_{h=1}^{H} e^{2\pi i h (x-y)} \right) dg(x) dg(y) = \iint_{\{(x,y) \in [0,1]^2 :\ x-y \in Z\}} dg(x) dg(x)$$

and the last integral is zero if and only if g is continuous. In particular, if (f(t)) has a continuous (M, λ)-a. w. c. d. f. (mod 1), then (3) follows. Finally, suppose that the limit (1) exists and that (3) holds. By the usual approximation methods, it follows that the limit

$$L(F)=\lim_{T\to\infty}\frac{1}{\Lambda(T,k)}\int_{0+k}^{T+k}\lambda(t)F(\{f(t)\})dt,$$

exists uniformly in $k\in[0,\infty)$ for every continuous function F on [0, 1] with $F(0)=F(1)$. If the space of these functions is equipped with the supremum norm, then L is a bounded linear functional on it with $L(F)\geqq 0$ whenever $F\geqq 0$. Thus, by the Riesz representation theorem

$$L(F)=\int_0^1 F(x)dg(x),$$

with a non-decresing function g on [0, 1]. Without loss of generality, we may assume $g(0)=0$. Then, by choosing $F=1$, we obtained $g(1)=1$.
By what we have already shown, g(x) is continuous. We have

$$\lim_{T\to\infty}\frac{1}{\Lambda(T,k)}\int_{0+k}^{T+k}\lambda(t)F(\{f(t)\})dt=\int_0^1 F(x)dg(x)=L(F),$$

uniformly in $k\in[0,\infty)$ where g(x) is continuous.
By Theorem 1, the proof is completed. (q.e.d.).

Proof of 3. The existence of the limit (5) is necessary. Hence from Parseval's theorem and by the assumption, we have

$$\sum_{h\in Z}|\alpha_h|^2<+\infty.$$

This proves the necessity. Next we have show the sufficiency. By Riesz-Fisher theorem, there exists $dg\in L^2(0,1)$ such that

$$\int_0^1 e^{2\pi inx}dg(x)=\alpha_n. \tag{7}$$

Since $dg\in L^2(0,1)\subset L(0,1)$ has a Fourier series that is dominatedly convergent almost everywhere, it follows, after correcting dg on a null set, that

$$dg(x)=\sum_{n\in Z}\alpha_n e^{2\pi inx} \quad \text{for all } x\in(0,1). \tag{8}$$

From (7), (8) and by Lebesgue's theorem on the derivative of integrals it follows that g(x) is absolutely continuous and $g'\in L^2(0,1)$. (q.e.d.).

References

[1] K. Goto and T. Kano: On the Wiener-Schoenberg Theorem of Asymptotic Distribution Functions. Proc. Japan Acad., vol. 57. 420-423 (1981).
[2] L. Kuipers and H. Niederreiter: Uniform distribution of sequences, Wiley, 1974.